NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1438

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By R. D. Stepanov

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ON THE FLUTTER OF CYLINDRICAL SHELLS AND PANELS MOVING IN A FLOW OF GAS*

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The equations of shells are taken in the form of the general technical theory of shallow shells (ref. 1) and shells of medium length of V. Z. Vlasov (ref. 2). The aerodynamic forces acting on a shell are taken into account only as forces of excess pressure according to the formula proposed by A. A. Iliushin (ref. 3). In this work the following notation is used: α and β are dimensionless coordinates of a point on the cylindrical surface of the shell. The coordinate α, expressed in terms of radius R, represents the distance along a generator, and β represents the central angle. The dimensions R, h, and l are the radius, thickness, and length of the cylindrical shell; E, σ, and ρ are Young's modulus, Poisson's coefficient, and the density of the material of the shell, and D is the cylindrical rigidity. The quantities u, v, and w are components of the vector of displacement of the shell, V is the velocity of flow, v_0 is the velocity of sound at infinity, ρ_0 is the pressure of the gaseous medium at infinity, and κ is the exponent of polytropy. The symbol $\omega = p + iq$ is the complex frequency, and c and B are constants. Also,

$$c^2 = \frac{h^2}{12R^2} \qquad D = \frac{Eh^3}{12\left(1 - \sigma^2\right)} \qquad B = \rho_0 \frac{\kappa}{v_0} \qquad \nabla^2 = \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}$$

The quantity B_1 is the coefficient of damping, and Z is the transverse component of the load.

1. INITIAL RELATIONS OF THE THEORY OF CYLINDRICAL SHELLS

In the system of dimensionless coordinates α,β, in the case when at each point a load directed along the normal to the surface acts on

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NACA Reviewer's note: The original Russian publication contains certain typographical errors and obvious omissions in equations that have been corrected without comment.

the shell $(X = Y = 0,\ \ Z \neq 0)$, the fundamental equation of shallow cylindrical shells in a form convenient for solution is (ref. 1):

$$\nabla^2 \nabla^2 \nabla^2 \nabla^2 \Phi + \frac{1 - \sigma^2}{c^2} \frac{\partial^4 \Phi}{\partial \alpha^4} = \frac{R^4}{D} Z \tag{1.1}$$

where $\Phi(\alpha,\beta)$ is a scalar function, defined according to the formulas

$$\left.\begin{aligned}
u &= \frac{\partial^3 \Phi}{\partial \alpha\, \partial \beta^2} - \sigma \frac{\partial^3 \Phi}{\partial \alpha^3} \\
v &= -\left[\frac{\partial^3 \Phi}{\partial \beta^3} + (2 + \sigma) \frac{\partial^3 \Phi}{\partial \alpha^2 \partial \beta}\right] \\
w &= \nabla^2 \nabla^2 \Phi
\end{aligned}\right\} \tag{1.2}$$

The internal forces of shallow cylindrical shells are defined through the function $\Phi(\alpha,\beta)$ by the following group of formulas:

$$\left.\begin{aligned}
N_1 &= \frac{Eh}{R} \frac{\partial^4 \Phi}{\partial \alpha^2 \partial \beta^2} & M_1 &= \frac{D}{R^2}\left(\frac{\partial^2}{\partial \alpha^2} + \sigma \frac{\partial^2}{\partial \beta^2}\right) \nabla^2 \nabla^2 \Phi \\
N_2 &= \frac{Eh}{R} \frac{\partial^4 \Phi}{\partial \alpha^4} & M_2 &= \frac{D}{R^2}\left(\frac{\partial^2}{\partial \beta^2} + \sigma \frac{\partial^2}{\partial \alpha^2}\right) \nabla^2 \nabla^2 \Phi \\
S &= -\frac{Eh}{R} \frac{\partial^4 \Phi}{\partial \alpha^3 \partial \beta} & H &= -\frac{D(1 - \sigma)}{R^2} \frac{\partial^2}{\partial \alpha\, \partial \beta} \nabla^2 \nabla^2 \Phi \\
Q_1 &= -\frac{D}{R^3} \frac{\partial}{\partial \alpha} \nabla^2 \nabla^2 \nabla^2 \Phi & Q_2 &= -\frac{D}{R^3} \frac{\partial}{\partial \beta} \nabla^2 \nabla^2 \nabla^2 \Phi
\end{aligned}\right\} \tag{1.3}$$

Generalized transverse forces defined in the sense of Kirchoff being necessary for the formulation of the boundary conditions, are computed according to the formulas

$$\left.\begin{aligned} Q_1{}^* &= -\frac{D}{R^3}\left[\frac{\partial^3}{\partial\alpha^3} + (2 - \sigma)\frac{\partial^3}{\partial\alpha\,\partial\beta^2}\right]\nabla^2\nabla^2\Phi \\ Q_2{}^* &= -\frac{D}{R^3}\left[\frac{\partial^3}{\partial\beta^3} + (2 - \sigma)\frac{\partial^3}{\partial\alpha^2\partial\beta}\right]\nabla^2\nabla^2\Phi \end{aligned}\right\} \qquad (1.4)$$

The positive directions of the forces and moments are shown in fig. 1.

In this work an approximate theory for the calculation of cylindrical shells of medium length (ref. 2) will be widely used.

In the system of dimensionless coordinates α,β, with $X = Y = 0$ but $Z \neq 0$, the fundamental equation of cylindrical shells of medium length, in the conveniently solvable form, is:

$$\frac{\partial^4\Phi_1}{\partial\alpha^4} + \frac{c^2}{1 - \sigma^2}\left[\frac{\partial^2}{\partial\beta^2} + 1\right]^2\frac{\partial^4\Phi_1}{\partial\beta^4} = \frac{R^2}{Eh}Z \qquad (1.5)$$

Here the function $\Phi_1(\alpha,\beta)$ is defined by the formulas

$$\left.\begin{aligned} u &= \frac{\partial^3\Phi_1}{\partial\alpha\,\partial\beta^2} \\ v &= -\frac{\partial^3\Phi_1}{\partial\beta^3} \\ w &= \frac{\partial^4\Phi_1}{\partial\beta^4} \end{aligned}\right\} \qquad (1.6)$$

The internal forces of cylindrical shells of medium length are expressed through the function $\Phi_1(\alpha,\beta)$ by the following group of formulas:

$$\left.\begin{aligned}
N_1 &= \frac{Eh}{R}\frac{\partial^4\Phi_1}{\partial\alpha^2\partial\beta^2} & N_2 &= \frac{Eh}{R}\left[\frac{\partial^4\Phi_1}{\partial\alpha^4} + \frac{c^2}{1-\sigma^2}\left(\frac{\partial^6\Phi_1}{\partial\beta^6} + \frac{\partial^4\Phi_1}{\partial\alpha^4}\right)\right] \\
M_1 &= \frac{D}{R^2}\sigma\left[\frac{\partial^6\Phi_1}{\partial\beta^6} + \frac{\partial^4\Phi_1}{\partial\beta^4}\right] & M_2 &= \frac{D}{R^2}\left[\frac{\partial^6\Phi_1}{\partial\beta^6} + \frac{\partial^4\Phi_1}{\partial\beta^4}\right] \\
S &= -\frac{Eh}{R}\frac{\partial^4\Phi_1}{\partial\alpha^3\partial\beta} & H &= -\frac{D}{R^2}(1-\sigma)\left[\frac{\partial^6\Phi_1}{\partial\alpha\,\partial\beta^5} + \frac{\partial^4\Phi_1}{\partial\alpha\,\partial\beta^3}\right] \\
Q_1 &= -\frac{D}{R^3}\left[\frac{\partial^7\Phi_1}{\partial\alpha\,\partial\beta^6} + \frac{\partial^5\Phi_1}{\partial\alpha\,\partial\beta^4}\right] & Q_2 &= -\frac{D}{R^3}\left[\frac{\partial^7\Phi_1}{\partial\beta^7} + \frac{\partial^5\Phi_1}{\partial\beta^5}\right]
\end{aligned}\right\} \tag{1.7}$$

In each particular case, it is necessary to adjoin given boundary conditions on the edges of the shell to the differential equations (1.1) or (1.5).

2. STATEMENT OF THE PROBLEM

The expression for the transverse load Z acting on an element of the surface of the shell is composed of two parts: the force of inertia

$$Z_1 = -\rho h\frac{\partial^2 w}{\partial t^2} \tag{2.1}$$

and the force of aerodynamic action of the supersonic flow directed along a generator and flowing around the shell on the outside, which is taken into account according to the formula proposed by A. A. Iliushin (ref. 3):

$$Z_2 = \frac{BV}{R}\frac{\partial w}{\partial\alpha} - B_1\frac{\partial w}{\partial t} \tag{2.2}$$

Substituting expression (2.2) into equation (1.1) and taking into account the third of the relations in the group (1.2) yields the differential equation of small vibrations of shallow cylindrical shells:

$$c_*^2\nabla^8\Phi + \frac{\partial^4\Phi}{\partial\alpha^4} + \frac{R^2}{E}\rho\frac{\partial^2}{\partial t^2}\nabla^4\Phi - \frac{BVR}{Eh}\frac{\partial}{\partial\alpha}\nabla^4\Phi + \frac{R^2B_1}{Eh}\frac{\partial}{\partial t}\nabla^4\Phi = 0 \tag{2.3}$$

Correspondingly, substituting the expression for the external load Z in equation (1.5) and taking into account the third relation of group (1.6) gives the equation of small vibrations of cylindrical shells of medium length:

$$\frac{\partial^4 \Phi_1}{\partial \alpha^4} + c_*^2\left[\frac{\partial^2}{\partial \beta^2} + 1\right]^2 \frac{\partial^4 \Phi_1}{\partial \beta^4} + \frac{R^2}{E}\rho \frac{\partial^6 \Phi_1}{\partial t^2 \partial \beta^4} - \frac{BVR}{Eh}\frac{\partial^5 \Phi_1}{\partial \alpha\, \partial \beta^4} + \frac{B_1 R^2}{Eh}\frac{\partial^5 \Phi_1}{\partial t\, \partial \beta^4} = 0 \qquad (2.4)$$

The new dimensionless quantity

$$c_*^2 = \frac{c^2}{1 - \sigma^2} = \frac{h^2}{12R^2(1 - \sigma^2)} \qquad (2.5)$$

is introduced in equations (2.3) and (2.4).

In all of the following calculations B_1 will be taken equal to B.

In 1954, under the guidance of A. A. Iliushin, an investigation was made (refs. 4 and 5) of the self-induced vibrations of a plate moving in a gas, which defined in many respects an approach and methods of solution of the problem set down; certain results borrowed from the indicated works will be introduced below without derivation.

Examined in this paper is a class of solutions of the form

$$\Phi(\alpha,\beta,t) = \psi(\alpha,\beta)e^{\omega t} \qquad (2.6)$$

where $\omega = p + iq$ is a constant complex frequency.

In the class of solutions (2.6), the problem of flutter consists in determining the least velocity of flow (this velocity will be called the critical velocity), which on being exceeded would result in a positive real part of the complex frequency.

Substituting equation (2.6) into equation (2.3) gives, after cancelling $e^{\omega t}$,

$$c_*^2 \nabla^8 \psi + \frac{\partial^4 \psi}{\partial \alpha^4} - \lambda \nabla^4 \psi - \frac{BVR}{Eh}\frac{\partial}{\partial \alpha}\nabla^4 \psi = 0 \qquad (2.7)$$

Here

$$-\lambda = \rho \frac{R^2 \omega^2}{E} + \frac{BR^2}{Eh}\omega \qquad (2.8)$$

Equation (2.8) permits us, for each particular eigenvalue λ, two values of frequency to be defined:

$$\omega_{1,2} = -\frac{B}{2\rho h} \pm \left[\left(\frac{B}{2\rho h}\right)^2 - \frac{E\lambda}{\rho R^2}\right]^{1/2} \tag{2.9}$$

For certain eigenvalues, let one of the roots of equation (2.9) be a pure imaginary number. Then from equation (2.8) it is easy to obtain

$$\left.\begin{aligned} \operatorname{Re} \lambda &= \lambda_1 = \rho \frac{R^2}{E} q^2 \\ \operatorname{Im} \lambda &= \lambda_2 = -\frac{BR^2}{Eh} q \end{aligned}\right\} \tag{2.10}$$

On the complex plane λ_1, λ_2, equation (2.10) represents the points of the square parabola (fig. 2)*

$$\lambda_1 = \rho \frac{Eh^2}{B^2R^2} \lambda_2^2 \tag{2.11}$$

which, following the example of references 4 and 5, is called the parabola of stability. The region lying inside the parabola of stability corresponds to the eigenvalues for which the roots (2.9) have a negative real part, while the region lying outside the parabola corresponds to eigenvalues for which the real parts of one of the roots (2.9) has a positive real part.

3. UNBOUNDED CLOSED CYLINDRICAL SHELL

For the case of an unbounded closed cylindrical shell, the solution of the fundamental differential equation of small vibrations (2.7) will be sought in the form

$$\psi(\alpha,\beta) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} C_{kn} e^{i(n\beta+k\alpha)} \tag{3.1}$$

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*Translator's note: This apparently refers to figure 2 of references 4 and 5 which is included with the figures of this paper for the convenience of the reader.

where $C_{k,n}$ is a certain constant number and n and k are constant numbers denoting the number of half waves in the meridional direction and in the direction of the generators of the shell.

Substituting equation (3.1) into equation (2.7), yields a characteristic equation from which the following expression for λ results:

$$\left.\begin{aligned} \lambda_1 &= c_*^2\left(k^2 + n^2\right)^2 + \frac{k^4}{\left(k^2 + n^2\right)^2} \\ \lambda_2 &= -\frac{BVR}{Eh}k \end{aligned}\right\} \tag{3.2}$$

On the complex plane λ_1,λ_2 equation (3.2) represents points of a parabola of the eighth power:

$$\lambda_1 = c_*^2\left[\frac{E^2h^2}{B^2R^2V^2}\lambda_2^2 + n^2\right]^2 + \frac{E^4h^4\lambda_2^4}{B^4V^4R^4}\left[\frac{E^2h^2}{B^2R^2V^2}\lambda_2^2 + n^2\right]^{-2} \tag{3.3}$$

For the determination of the critical velocity of flow, an analysis is made of the problem of the mutual arrangement of the parabola (3.3) with the parabola of stability (2.11) in the cases $n = 0$ and $n \neq 0$. For $n = 0$ (that is, for the case when the contour of a transverse section of the shell remains a circle in the process of deformation) equations (3.2) take the form:

$$\left.\begin{aligned} \lambda_1 &= c_*^2k^4 + 1 \\ \lambda_2 &= -\frac{BRV}{Eh}k \end{aligned}\right\} \tag{3.4}$$

For the points of mutual intersection of the parabola (3.4) with the parabola of stability, the equalities

$$\left.\begin{aligned} \rho\frac{R^2q^2}{E} &= c_*^2k^4 + 1 \\ \frac{BR^2}{Eh}q &= \frac{BVR}{Eh}k \end{aligned}\right\} \tag{3.5}$$

are valid.

Eliminating from the first of equations (3.5) the parameter q yields one equation for the determination of the points of mutual intersection of the two parabolas being investigated:

$$k^4 - \rho \frac{V^2}{Ec_*^2} k^2 + \frac{1}{c_*^2} = 0 \tag{3.6}$$

the solution of which will be

$$k_{1,2,3,4} = \pm\left\{\frac{\rho V^2}{2Ec_*^2} \pm \left[\left(\frac{\rho V^2}{2Ec_*^2}\right)^2 - \frac{1}{c_*^2}\right]^{1/2}\right\}^{1/2} \tag{3.7}$$

From equation (3.7), it follows that, for

$$V_* > \left(\frac{2Ec_*}{\rho}\right)^{1/2} \tag{3.8}$$

the parabola (3.4), intersecting with the parabola of stability in four points, extends outside the domain of stability. Hence, it follows that for a velocity of flow larger than $(2Ec_*/\rho)^{1/2}$, the motion of the shell must be unstable.

For the investigation of the problem of the mutual intersection of the parabola of stability with the parabola (3.3), in the general case for $n \neq 0$ the following equation is obtained:

$$k^8 + k^6\left(4n^2 - \rho \frac{V^2}{Ec_*^2}\right) + k^4\left(6n^4 + \frac{1}{c_*^2} - 2\rho \frac{V^2n^2}{Ec_*^2}\right) +$$
$$k^2\left(4n^6 - \rho n^4 \frac{V^2}{Ec_*^2}\right) + n^8 = 0 \tag{3.9}$$

Solution of equation (3.9) gives the eight roots:

$$k_i = \pm 2n^2 \left\{\left[-a \pm \left(a^2 - 4b + 8n^4\right)^{1/2}\right] \pm \right.$$
$$\left.\left[\left(-a \pm \left[a^2 - 4b + 8n^4\right]^{1/2}\right)^2 - 16n^4\right]\right\}^{-1/2} \tag{3.10}$$

where

$$\left.\begin{aligned} a &= 4n^2 - \rho \frac{V^2}{Ec_*^2} \\ b &= 2n^2 a + \frac{1}{c_*^2} - 2n^4 \end{aligned}\right\} \tag{3.11}$$

Similarly to that which was done above for the case $n = 0$, it is possible here also to show that the necessary and sufficient conditions for which the parabola (3.3) intersects with the parabola of stability, and hence falls outside the domain of stability, reduce to the determination of the condition of the appearance of complex roots of equation (3.10).

Analyzing the expression (3.10), it is possible to set down the two following essentially distinct necessary and sufficient conditions that the parabola (3.3) intersect with the parabola of stability, and hence, extend beyond the domain of stability:

$$a^2 - 4b + 8n^4 = \rho^2 \frac{V^4}{E^2 c_*^4} - \frac{4}{c_*^2} \geqq 0, \qquad -a \pm \left(a^2 - 4b + 8n^4\right)^{1/2} < 4n^2 \tag{3.12}$$

In expressions (3.12) it is necessary to satisfy the inequality

$$a = 4n^2 - \rho \frac{V^2}{Ec_*^2} < 0 \tag{3.13}$$

The inequalities (3.12) and (3.13) make it possible to determine the critical velocities:

$$V_* \geqq \left(\frac{2Ec_*}{\rho}\right)^{1/2}, \qquad n < \frac{1}{2}\left(c_*\right)^{-1/2} = n_* \tag{3.14}$$

$$V_* \geqq \frac{1}{2n}\left[\frac{E}{\rho}\left(16n^4 c_*^2 + 1\right)\right]^{1/2} \tag{3.15}$$

The formula for the critical velocity (3.14) identically coincides with the critical velocity of flow found for a closed cylindrical shell with $n = 0$, and, as is seen from the inequality (3.13), it may be used for all values $n < n_*$ which, for thin shells, corresponds to the number

of half waves $n \approx 30$ to 50, that is, to such a large number of half-waves that the form of the transverse section differs little from a circle.

The minimum velocity (3.15), according to n, takes place for $n = \frac{1}{2} c_*^{-1/2}$ and exactly coincides with the critical velocity found above for $n = 0$.

From this analysis it follows that flutter of a closed cylindrical shell of infinite length in a supersonic flow can possibly take place for velocities of flow $V > (2Ec_*/\rho)^{1/2}$ when the form of the transverse section remains circular.

Using the formulas (2.9) and (3.2), it is possible to obtain two values of frequency whose essential form depends on the velocity of flow:

$$\omega_{1,2} = -\frac{B}{2\rho h} \pm \left\{\left(\frac{B}{2\rho h}\right)^2 - \frac{E}{\rho R^2}\left[c_*^2(k^2 + n^2)^2 + \frac{k^4}{(k^2 + n^2)^2}\right] + \frac{BVk}{\rho h} i\right\}^{1/2} \tag{3.16}$$

The solution of the differential equation of small vibrations of shallow shells (2.3), taken in the form

$$\Phi(\alpha,\beta,t) = e^{i(n\beta+k\alpha)}e^{(p+iq)t} \tag{3.17}$$

signifies that along the generators of the shell are propagated waves traveling with the velocity

$$v_b = -\frac{q}{k} \tag{3.18}$$

Separating the real part of the complex frequency (eq.(3.16)) from the imaginary part yields

$$v_b = \pm\left\{\frac{1}{2k^2}\left[\pm\left\{\left[\left(\frac{B}{2\rho h}\right)^2 - \frac{E}{\rho R^2}\left(c_*^2(k^2 + n^2)^2 + \frac{k^4}{(k^2 + n^2)^2}\right)\right]^2 + \frac{B^2V^2k^2}{h^2\rho^2R^2}\right\}^{1/2} - \left[\frac{B^2}{4\rho^2h^2} - \frac{E}{\rho R^2}\left(c_*^2(k^2 + n^2)^2 + \frac{k^4}{(k^2 + n^2)^2}\right)\right]\right]\right\}^{1/2} \tag{3.19}$$

Using formula (3.19), the velocity of propagation of a traveling wave is determined for $V = 0$:

$$v_b = \pm\left\{\frac{E}{\rho R^2}\left[c_*^2\,\frac{\left(k^2 + n^2\right)^2}{k^2} + \frac{k^2}{\left(k^2 + n^2\right)^2} + \frac{B^2}{4\rho^2 h^2 k^2}\right]\right\}^{1/2}$$

The minimum velocity of propagation of a traveling wave occurs for

$$n^2 = k\left[\left(\frac{1}{c_*}\right)^{1/2} - k\right] \tag{3.20}$$

Here, it has the value

$$v_{b_{min}} = \left[\frac{2Ec_*^2}{\rho R^2} - \frac{B^2}{4\rho^2 h^2 k^2}\right]^{1/2} \tag{3.21}$$

4. UNBOUNDED CYLINDRICAL PANEL, SIMPLY SUPPORTED ALONG ITS GENERATORS

In the case of an unbounded cylindrical panel, simply supported along its generators, the differential equations of small vibrations (2.3) must be accompanied by the boundary conditions:

$$u = w = 0 \qquad N_2 = M_2 = 0 \qquad \text{at} \quad \beta = 0 \quad \text{and} \quad \beta = \beta_1 = s/R$$

Defining, according to formulas (1.2) and 1.3), the displacements and internal forces of the shell through a potential function Φ, it is possible to write the boundary conditions on the edges $\beta = 0$, $\beta = \beta_1$ in the form

$$\Phi = \frac{\partial^2\Phi}{\partial\beta^2} = \frac{\partial^4\Phi}{\partial\beta^4} = \frac{\partial^6\Phi}{\partial\beta^6} = 0 \tag{4.2}$$

Representing the solution of the indicated boundary-value problem in the form

$$\Phi(\alpha,\beta,t) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} A_{kn} \sin \frac{n\pi\beta}{\beta_1} e^{ik\alpha} e^{\omega t} \tag{4.3}$$

and producing calculations analogous to those which were mentioned above for the general case $n \neq 0$, gives the formula for the critical velocity of flow, as follows:

$$V_* \geqq \frac{\beta_1}{2n\pi}\left[\frac{E}{\rho}\left(16 \frac{n^4\pi^4}{\beta_1{}^4} c_*{}^2 + 1\right)\right]^{1/2} = \left[\frac{E}{\rho}\left(\frac{4n^2\pi^2h^2}{12s^2\left(1 - \sigma^2\right)} + \frac{s^2}{4R^2n^2\pi^2}\right)\right]^{1/2} \tag{4.4}$$

At $R \rightarrow \infty$,

$$V_{*min} \geqq \frac{\pi h}{s}\left[\frac{E}{\rho} \frac{1}{3\left(1 - \sigma^2\right)}\right]^{1/2} \quad \text{at} \quad n = 1 \tag{4.5}$$

Formula (4.5) coincides with the critical velocity of flow for an infinite plate, simply supported along the edge parallel to the direction of flow, which was obtained first in reference 4.

*Omitting all intermediate calculations, we quote the formula of the critical velocity of flow for the unbounded closed cylindrical shell, found from consideration of the differential equation of small vibrations of cylindrical shells of medium length (2.4):

$$V_* \geqq \left[\frac{2E}{\rho} c_*\left(1 - \frac{1}{n^2}\right)\right]^{1/2} \tag{4.6}$$

It is possible to make use of formula (4.6) for all values of $n \geqq 2$. From this formula it follows, that for $n = \infty$, the critical velocity of an unbounded closed cylindrical shell of medium length coincides with the velocity of the unbounded closed cylindrical shell found starting from the theory of shallow shells.

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*Translator's note: This paragraph and the one following seem misplaced. They were probably intended to conclude the preceding section.

5. FLUTTER OF CLOSED CYLINDRICAL SHELLS OF BOUNDED LENGTH FOR DIFFERENT BOUNDARY CONDITIONS ON THE ENDS

The equation of small vibrations of shells of medium length (2.4) is used to examine a series of boundary problems. In this examination is introduced a new variable ξ, connected with α by the formula

$$\alpha = \frac{l}{R}\xi \tag{5.1}$$

Then in a form convenient for solution, the equation of small vibration (2.4) is written

$$\frac{\partial^4\Phi_1}{\partial\xi^4} + c_*{}^2\frac{l^4}{R^4}\left(\frac{\partial^2}{\partial\beta^2}+1\right)^2\frac{\partial^4\Phi_1}{\partial\beta^4} + \rho\frac{l^4}{ER^2}\frac{\partial^6\Phi_1}{\partial t^2\partial\beta^4} + \frac{Bl^4}{EhR^2}\frac{\partial^5\Phi_1}{\partial t\,\partial\beta^4} - \frac{BVl^3}{EhR^2}\frac{\partial^5\Phi_1}{\partial\xi\,\partial\beta^4} = 0 \tag{5.2}$$

To equation (5.2), in each particular case, must be added boundary conditions on the ends $\xi = 0$ and $\xi = 1$.

By defining, according to formulas (1.6) and (1.7), the displacements and internal forces of the shell through Φ_1, the boundary conditions for the boundary problems may be represented in the following forms:

(a) For a shell simply supported on the ends $\xi = 0, \xi = 1$:

$$w = \frac{\partial^4\Phi_1}{\partial\beta^4} = 0 \qquad M_1 = \frac{D}{R}\sigma\left[\frac{\partial^6\Phi_1}{\partial\beta^6} + \frac{\partial^4\Phi_1}{\partial\beta^4}\right] = 0 \qquad \text{at } \xi = 0 \text{ and } \xi = 1 \tag{5.3}$$

(b) For a shell clamped on the ends $\xi = 0$ and $\xi = 1$:

$$w = \frac{\partial^4\Phi_1}{\partial\beta^4} = 0 \qquad \frac{\partial w}{\partial\xi} = \frac{R}{l}\frac{\partial^5\Phi_1}{\partial\xi\,\partial\beta^4} = 0 \qquad \text{at } \xi = 0 \text{ and } \xi = 1 \tag{5.4}$$

(c) For a shell simply supported on the end $\xi = 1$ and rigidly clamped on the edge $\xi = 0$:

$$\left.\begin{aligned} &w = \frac{\partial^4\Phi_1}{\partial\beta^4} = 0 \qquad \frac{\partial w}{\partial\xi} = \frac{R}{l}\frac{\partial^5\Phi_1}{\partial\xi\,\partial\beta^4} = 0 \qquad \text{at} \quad \xi = 0 \\ &w = \frac{\partial^4\Phi_1}{\partial\beta^4} = 0 \qquad M_1 = \frac{D}{R}\,\sigma\left[\frac{\partial^6\Phi_1}{\partial\beta^6} + \frac{\partial^4\Phi_1}{\partial\beta^4}\right] = 0 \qquad \text{at} \quad \xi = 1 \end{aligned}\right\} \tag{5.5}$$

(d) For a shell clamped on the end $\xi = 0$ and force on the end $\xi = 1$:

$$\left.\begin{aligned} &w = \frac{\partial^4\Phi_1}{\partial\beta^4} = 0 \qquad \frac{\partial w}{\partial\xi} = \frac{R}{l}\frac{\partial^5\Phi_1}{\partial\xi\,\partial\beta^4} = 0 \qquad \text{at} \quad \xi = 0 \\ &N_1 = \frac{EhR}{l^2}\frac{\partial^4\Phi_1}{\partial\xi^2\partial\beta^2} = 0 \qquad S = \frac{EhR^2}{l^3}\frac{\partial^4\Phi_1}{\partial\xi^3\partial\beta} = 0 \qquad \text{at} \quad \xi = 1 \end{aligned}\right\} \tag{5.6}$$

(From the second group of relations it is seen that the boundary conditions on the free end are partially satisfied.)

(e) For a shell simply supported on the end $\xi = 0$ and free on the edge $\xi = 1$:

$$\left.\begin{aligned} &w = \frac{\partial^4\Phi_1}{\partial\beta^4} = 0 \qquad M_1 = \frac{D}{R}\,\sigma\left[\frac{\partial^6\Phi_1}{\partial\beta^6} + \frac{\partial^4\Phi_1}{\partial\beta^4}\right] = 0 \qquad \text{at} \quad \xi = 0 \\ &N_1 = \frac{EhR}{l^2}\frac{\partial^4\Phi_1}{\partial\xi^2\partial\beta^2} = 0 \qquad S = \frac{EhR^2}{l^3}\frac{\partial^4\Phi_1}{\partial\xi^3\partial\beta} = 0 \qquad \text{at} \quad \xi = 1 \end{aligned}\right\} \tag{5.7}$$

For the class of solutions

$$\Phi_1(\xi,\beta,t) = \sum_{n=2}^{\infty}\sum_{k=1}^{\infty} C_{kn}X_k(\xi)e^{\omega t}\cos n\beta \tag{5.8}$$

equation (5.2) is written after a series of simple transformations, in the form

$$\frac{d^4X_k}{d\xi^4} - An^4\frac{dX_k}{d\xi} + \left[c_1^2(n^2 - 1)^2 n^4 - \lambda n^4\right]X_k = 0 \tag{5.9}$$

where

$$
\left.
\begin{aligned}
-\lambda &= \frac{l^4}{ER^2}\left[\rho\omega^2 + \frac{B}{h}\,\omega\right] \\
A &= \frac{Bl^3}{EhR^2}\,V \\
c_1{}^2 &= c_*{}^2\,\frac{l^4}{R^4} = \frac{h^2 l^4}{12R^6\left(1 - \sigma^2\right)}
\end{aligned}
\right\} \qquad (5.10)
$$

The equation of the parabola of stability will have the form:

$$
\lambda_1 = \rho\,\frac{h^2ER^2}{B^2 l^4}\,\lambda_2{}^2 \qquad \left(\lambda_1 = \frac{l^4}{ER^2}\,\rho q^2, \quad \lambda_2 = -\frac{Bl^4}{EhR^2}\,q\right) \qquad (5.11)
$$

For fixed c_1, n, A, and λ the solution of equation (5.9), in the case when the roots of the characteristic equation are distinct, has the form

$$
X_k(\xi) = C_1e^{-k_1\xi} + C_2e^{-k_2\xi} + C_3e^{-k_3\xi} + C_4e^{-k_4\xi} \qquad (5.12)
$$

The rest of the problem reduces to the determination of nontrivial solutions C_i, for which it is sufficient to subject the solution (eq. (5.12)) to the boundary conditions and to require the vanishing of the appropriate determinant $\Delta(k_i)$. Avoiding the problem of the form of the determinant $\Delta(k_i)$ for the different possible combinations of multiple roots, the function

$$
F(k_i) = \frac{\Delta(k_i)}{\delta(k_i)} \qquad (5.13)
$$

is introduced where

$$
\delta(k_i) = (k_1 - k_2)(k_1 - k_3)(k_1 - k_4)(k_2 - k_3)(k_2 - k_4)(k_3 - k_4)
$$

From the expression $\delta(k_i)$ it follows that all zeros of the functions $\Delta(k_i)$ will exist by virtue of zeros of $\delta(k_i)$, and $F(k_i)$ will be an analytic function in the whole domain of the variation of the variables.

The solution of equation (5.9) in its general case is accompanied with considerable mathematical difficulties, and for this reason we apply here the method of investigation of the eigenvalues λ which was proposed in references 4 and 5. The essence of the method is contained in that, instead of the solution of equation (5.9), the parameters of the problem A, λ and the two sought for roots k_3, k_4 are expressed through two other roots k_1, k_2 of the equations

$$\left.\begin{aligned} A &= -\frac{1}{n^4}\left[k_2^3 + k_2^2 k_1 + k_1^2 k_2 + k_1^3\right] \\ \lambda &= c_1^2\left(n^2 - 1\right)^2 - \frac{k_1 k_2 k_3 k_4}{n^4} \\ k_{3,4} &= -\frac{k_1 + k_2}{2} \pm \left[k_1 k_2 - \frac{3}{4}(k_1 + k_2)^2\right]^{1/2} \end{aligned}\right\} \qquad (5.14)$$

and instead of finding the eigenvalues of equation (5.9) there is investigated a system of two equations comprising the characteristic system

$$\left.\begin{aligned} A + \frac{4\eta}{n^4}\left(\eta^2 - \gamma^2\right) &= 0 \\ F(\eta,\gamma) = \frac{\Delta(\eta,\gamma)}{\delta(\eta,\gamma)} &= 0 \end{aligned}\right\} \qquad (5.15)$$

where η and γ are quantities connected with the roots

$$k_1 = \eta + i\gamma \qquad k_2 = \eta - i\gamma \qquad (5.16)$$

of the equation and

$$\delta(\eta,\gamma) = 16i\gamma\left[\gamma^2 - 2\eta^2\right]^{1/2}\left[\left(\gamma^2 - 3\eta^2\right) + 4\eta^2\gamma^2\right] \qquad (5.17)$$

The left part of each of equations (5.15) represents an analytic function of the variables η and γ, and the problem consists in finding a solution such as

$$\eta_i = \eta_i(n,A) \qquad \gamma_i = \gamma_i(n,A) \qquad (5.18)$$

of the system, which permits, through use of the formulas

$$\left.\begin{aligned} A &= -\frac{4\eta}{n^4}\left(\eta^2 - \gamma^2\right) \\ k_{3,4} &= -\eta \pm \left[\gamma^2 - 2\eta^2\right]^{1/2} \\ \lambda &= c_1{}^2\left(\eta^2 - 1\right)^2 + \frac{\eta^2 + \gamma^2}{n^4}\left(\gamma^2 - 3\eta^2\right) \end{aligned}\right\} \qquad (5.19)$$

for each boundary problem, to compute the corresponding eigenvalues λ and to establish that value A at which an eigenvalue becomes complex.

The solution of the characteristic system is always easier to obtain graphically if graphs of the curves defining the equations (5.15) are penciled on one sheet in a rectangular system of coordinates η,γ. The general form of the curves of the characteristic system is shown in figure 2; the graphs of the curves corresponding to the first equation of the system (hyperbolas) are drawn at different values A = Constant. The rest of the problem reduces to the establishment of those values A_{*i} where points of the first and second real branches (5.18) coincide, and it is impossible to make any deduction concerning the eigenvalues of the examined boundary problems.

Equating $A = A_{*i}$ according to equations (5.10), the velocity of flow at which there still exists stability of the unperturbed motion, but above which the motion may possibly become unstable is found. Consequently, for each particular boundary problem it is necessary, first of all, to compose the expression for the second equation of the characteristic system $\Delta(\eta,\gamma) = 0$.

The composition of the determinant $\Delta(\eta,\gamma)$ is shown for the example of a simply supported shell. For the determination of nonzero $C_i (i = 1, 2, 3, 4)$, expression (5.12) for $X_k(\xi)$ is subjected to the boundary conditions (5.3) and the determinant of the system thus obtained is equated to zero:

$$\Delta(k_1,k_2,k_3,k_4) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ k_1{}^2 & k_2{}^2 & k_3{}^2 & k_4{}^2 \\ e^{-k_1} & e^{-k_2} & e^{-k_3} & e^{-k_4} \\ k_1{}^2e^{-k_1} & k_2{}^2e^{-k_2} & k_3{}^2e^{-k_3} & k_4{}^2e^{-k_4} \end{vmatrix} = 0$$

The determinant having been disclosed, producing in it a replacement of k_i in terms of η and γ according to the formulas (5.16) results in

(a)

$$\Delta(\eta,\gamma) = \left\{-2\eta^2\gamma\left[\gamma^2 - 2\eta^2\right]^{1/2}\cosh 2\eta + 2\eta^2\gamma\left[\gamma^2 - 2\eta^2\right]^{1/2}\cos \gamma^2\cosh\left[\gamma^2 - 2\eta^2\right]^{1/2} - \left(3\eta^4 - \gamma^2 - 2\eta^2\gamma^2\right)\sin \gamma \sinh\left[\gamma^2 - 2\eta^2\right]^{1/2}\right\}16i = 0 \qquad (5.20)$$

The expression $\Delta(\eta,\gamma)$ for different boundary problems is obtained by analogous means:

(b) For a shell clamped on the ends $\xi = 0$ and $\xi = 1$,

$$\Delta(\eta,\gamma) = 8i\left\{\gamma\left[\gamma^2 - 2\eta^2\right]^{1/2}\left[\cos \gamma \cosh\left(\gamma^2 - 2\eta^2\right)^{1/2} - \cosh 2\eta\right] + 3\eta^2\sin \gamma \sinh\left[\gamma^2 - 2\eta^2\right]^{1/2}\right\} = 0 \qquad (5.21)$$

(c) For a shell clamped on the end $\xi = 0$ and simply supported on the end $\xi = 1$,

$$\Delta(\eta,\gamma) = 8i\left\{2\eta\gamma\left[\gamma^2 - 2\eta^2\right]^{1/2}\sinh 2\eta + \left(\gamma^2 - 3\eta^2\right)\left[\gamma^2 - 2\eta^2\right]^{1/2}\sin \gamma \cosh\left[\gamma^2 - 2\eta^2\right]^{1/2} - \gamma\left(\gamma^2 + \eta^2\right)\cos \gamma \sinh\left[\gamma^2 - 2\eta^2\right]^{1/2}\right\} = 0 \qquad (5.22)$$

(d) For a shell clamped on the end $\xi = 0$ and free on the end $\xi = 1$,

$$\Delta(\eta,\gamma) = i\left\{8\gamma(\eta^2 + \gamma^2)^2\left[\gamma^2 - 2\eta^2\right]^{1/2}\cosh 2\eta + \right.$$

$$4\gamma(26\eta^4 + 2\gamma^4 - 4\eta^2\gamma^2)\left[\gamma^2 - 2\eta^2\right]^{1/2}\cos\gamma . \cosh\left[\gamma^2 - 2\eta^2\right]^{1/2} +$$

$$8\eta^2(2\eta^2\gamma^2 - \gamma^4 + 3\eta^4)\sin\gamma \sinh\left[\gamma^2 - 2\eta^2\right]^{1/2} -$$

$$16\eta\gamma(\gamma^4 - \eta^4)\cos\gamma \sinh\left[\gamma^2 - 2\eta^2\right]^{1/2} -$$

$$16\eta(4\eta^2\gamma^2 - 3\eta^4 - \gamma^4)\left[\gamma^2 - 2\eta^2\right]^{1/2}\sin\gamma \cosh\left[\gamma^2 - 2\eta^2\right]^{1/2} -$$

$$\left. 32\eta^2\gamma^2(\gamma^2 - \eta^2)\left[\gamma^2 - 2\eta^2\right]^{1/2} e^{-2\eta}\right\} = 0 \qquad (5.23)$$

(e) For a shell simply supported on the end $\xi = 0$ and free on the end $\xi = 1$,

$$\Delta(\eta,\gamma) = i\left\{-2\eta\gamma(\gamma^2 + \eta^2)\left[\gamma^2 - 2\eta^2\right]^{1/2}\cosh 2\eta + \right.$$

$$\eta\gamma(\gamma^2 - 2\eta^2)^{1/2}\left[(\gamma^2 - \eta^2)^2 + (\gamma^2 - 3\eta^2)^2\right]e^{-2\eta} +$$

$$8\eta^3\gamma\left[\gamma^2 - 2\eta^2\right]^{1/2}(\gamma^2 - \eta^2)\cos\gamma \cosh\left[\gamma^2 - 2\eta^2\right]^{1/2} +$$

$$4\eta(3\eta^2\gamma^4 - \gamma^6 + 3\eta^6 - 5\eta^4\gamma^2)\sin\gamma \sinh\left[\gamma^2 - 2\eta^2\right]^{1/2} +$$

$$\gamma\left[5\eta^2\gamma^4 - \gamma^6 - 19\eta^4\gamma^2 + 23\eta^6\right]\cos\gamma \sinh\left[\gamma^2 - 2\eta^2\right]^{1/2} +$$

$$\left.\left[\gamma^2 - 2\eta^2\right]^{1/2}(\gamma^6 + 11\eta^4\gamma^2 - \eta^2\gamma^4 - 3\eta^6)\sin\gamma \cosh\left[\gamma^2 - 2\eta^2\right]^{1/2}\right\} = 0$$

$$(5.24)$$

It is noted that, for $\eta = 0$, the equation $\Delta(\eta,\gamma) = 0$ degenerates into the characteristic equations of the beam functions fundamental for the corresponding boundary problems.

For all the considered boundary problems, the graphs of equations (5.15) were constructed. With the help of these, the critical velocities of flow (fig. 2), that is, the velocities above which the unperturbed motion of the shell becomes unstable, were determined. Values of the critical velocities for three boundary problems (shells, simply supported on two sides, clamped on two sides, and clamped on one and simply supported on the other side) are quoted in table 1, for different ratios R/l and h/R.

Computation of the critical velocities for the cases shown were conducted for $n = 4$ at the following values of the constants:

$$E = 2 \times 10^{12} \frac{\text{dynes}}{\text{cm}^2} \qquad \rho = 7.8 \frac{\text{g}}{\text{cm}^3} \qquad \sigma = 0.3$$

$$p_0 = 1.014 \times 10^6 \frac{\text{dynes}}{\text{cm}^3} \qquad v_0 = 3.4 \times 10^4 \frac{\text{cm}}{\text{sec}} \qquad \kappa = 1.4$$

Here will be considered first the case where, for the class of thin shells of short and medium length, four transverse half-waves correspond to the fundamental mode of free vibrations of the one longitudinal half-wave. (The circle passes to an ellipse.)

It is possible to establish this situation if the frequency of free vibrations of closed cylindrical shells for the case of hinged support on the edges is investigated.

The minimum frequency of free vibrations of a simply supported closed cylindrical shell, calculated according to the theory of shallow shells, occurs at conditions coinciding with conditions (3.21), at which occurs the minimum velocity of propagation of a traveling wave along the generators of the shell in the absence of flow.

Because one of the equations of the characteristic system $F(\eta,\gamma) = 0$ does not depend on n, then from the first of equations (5.18) it follows on first glance that, for other conditions equal, with increase of the number of terms of the expansion n, self-induced vibration can possibly occur at lower velocities of flow. It is seen that there exist waves, the velocity of propagation of which (for all n, larger or smaller than $n = 4$) is above the velocity of propagation of the traveling wave occurring for $n = 4$. This situation has been successfully corroborated only for the case where the velocity of flow equals zero.

6. APPLICATION OF THE METHOD OF BUBNOV-GALERKIN TO THE INVESTIGATION OF SELF-INDUCED VIBRATIONS OF CYLINDRICAL SHELLS

In the preceeding paragraph a method was presented for the investigation of the eigenvalues of boundary problems resulting from the equation of small vibrations of cylindrical shells of medium length, which permits the determination of exact values of the critical velocities of flow.

An analogous method could not be successfully applied to the investigation of the eigenvalues of the separate boundary problems using the general equation of small vibrations of cylindrical shells (2.3). Therefore, for consideration of problems on self-induced-vibrations of closed cylindrical shells and cylindrical panels according to the theory of curved shells, a variational method was applied. First, there were determined by the variational method, the critical velocities of flow for the class of closed cylindrical shells of medium length with the different boundary conditions on the ends.

The values of the critical velocities resulting from the equation of shells of medium length, having been found in the second and third approximations, are given in table 1. From the table it is seen that the second approximation according to Galerkin gives a somewhat lower value of the critical velocity and the third approximation a somewhat higher value of the velocity as compared with the exact value of V_*; that is, the second and third approximations bracket the exact value of the critical velocity.

The good convergence of the variational method permitted its application to the investigation of problems of the self-induced vibration of closed cylindrical shells and cylindrical panels using the more general equation of shallow shells (eq. (2.3)).

In table 2 are quoted values of the critical velocities for a closed cylindrical shell simply supported on the ends, having been found starting from the theory of curved shells. The information given in table 2 with the corresponding magnitudes of the critical velocities quoted in table 1 makes it possible to remark that the difference in magnitude of the critical velocities does not exceed 10 to 15 percent and, consequently, for practical purposes it is entirely justified for the investigation of problems of the self-induced vibration of closed cylindrical shells to make use of the simple equation of shells of medium length.

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National Advisory Committee
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TABLE I.- VALUE OF THE CRITICAL VELOCITIES (m/sec) OF FLOW FOR CLOSED CYLINDRICAL SHELLS

[Shells simply supported on the ends (I); clamped on one end and simply supported on the other (II); and clamped on both ends (III)]

$\frac{h}{R}$	$\frac{R}{l}$	Exact			By the variational method					
					2nd approximation			3rd approximation		
		I	II	III	I	II	III	I	II	III
$\frac{1}{200}$	$\frac{1}{6}$	14,067	19,182	26,642	12,278	16,883	22,132	14,770	20,448	29,040
	$\frac{1}{8}$	5,934	9,092	11,239	5,203	7,141	9,352	6,332	8,869	11,880
	$\frac{1}{10}$	3,038	4,143	5,754	2,708	3,686	4,812	3,369	4,238	6,134
	$\frac{1}{12}$	1,758	2,397	3,330	1,648	2,186	2,822	1,999	2,656	3,526
$\frac{1}{300}$	$\frac{1}{6}$	9,378	12,788	17,761	8,197	11,266	14,765	9,847	13,939	19,360
	$\frac{1}{8}$	3,956	5,396	7,493	3,484	4,775	6,249	4,308	5,913	7,920
	$\frac{1}{10}$	2,025	2,769	3,836	1,687	2,475	3,225	2,246	3,314	4,089
	$\frac{1}{12}$	1,172	1,598	2,220	1,120	1,444	1,902	1,390	2,014	2,539
$\frac{1}{400}$	$\frac{1}{6}$	7,033	9,591	13,321	6,160	8,461	11,085	7,385	10,723	14,879
	$\frac{1}{8}$	2,968	4,048	5,623	2,629	3,596	4,701	3,233	4,437	5,943
	$\frac{1}{10}$	1,520	2,072	2,879	1,378	1,874	2,436	1,685	2,533	3,069
	$\frac{1}{12}$	879	1,199	1,666	861	1,130	1,447	1,043	1,626	1,976
$\frac{1}{500}$	$\frac{1}{6}$	5,626	7,673	10,656	4,940	6,781	8,880	5,908	8,578	11,903
	$\frac{1}{8}$	2,375	3,238	4,498	2,119	2,892	3,776	2,586	3,549	4,754
	$\frac{1}{10}$	1,216	1,658	2,303	1,129	1,512	1,968	1,348	2,026	2,595
$\frac{1}{750}$	$\frac{1}{6}$	3,751	5,115	7,104	3,322	4,548	5,946	3,938	5,718	7,367
	$\frac{1}{8}$	1,583	2,159	2,999	1,450	1,964	2,552	1,890	2,366	3,169
	$\frac{1}{10}$	810	1,105	1,535	796	1,056	1,354	1,025	1,350	1,636

TABLE II.- VALUES OF CRITICAL VELOCITIES OF FLOW FOR CLOSED CYLINDRICAL SHELLS, SIMPLY SUPPORTED ON THE ENDS, FOUND BY THE VARIATIONAL METHOD

$\frac{h}{R}$	$\frac{R}{l}$	By 2nd approximation m/sec	By 3rd approximation m/sec
$\frac{1}{200}$	$\frac{1}{6}$	10,858	14,770
	$\frac{1}{8}$	4,939	6,498
	$\frac{1}{10}$	2,684	3,463
	$\frac{1}{12}$	1,689	2,233
$\frac{1}{300}$	$\frac{1}{6}$	7,185	9,847
	$\frac{1}{8}$	3,261	4,332
	$\frac{1}{10}$	1,770	2,208
	$\frac{1}{12}$	1,117	1,452
$\frac{1}{400}$	$\frac{1}{6}$	5,384	7,385
	$\frac{1}{8}$	2,449	3,167
	$\frac{1}{10}$	1,337	1,732
$\frac{1}{500}$	$\frac{1}{6}$	4,313	5,908
	$\frac{1}{8}$	1,971	2,456
	$\frac{1}{10}$	1,086	1,268
	$\frac{1}{12}$	700	936
$\frac{1}{750}$	$\frac{1}{6}$	2,900	3,938
	$\frac{1}{8}$	1,348	1,775
	$\frac{1}{10}$	765	1,050

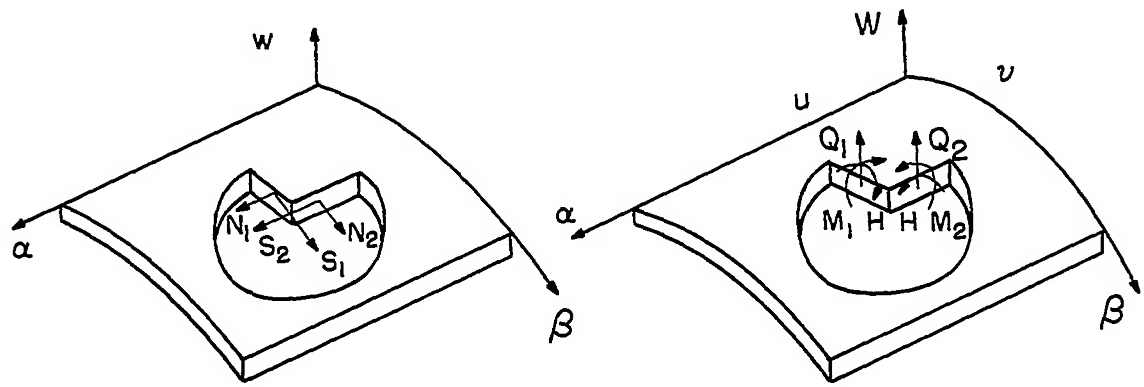

Figure 1.

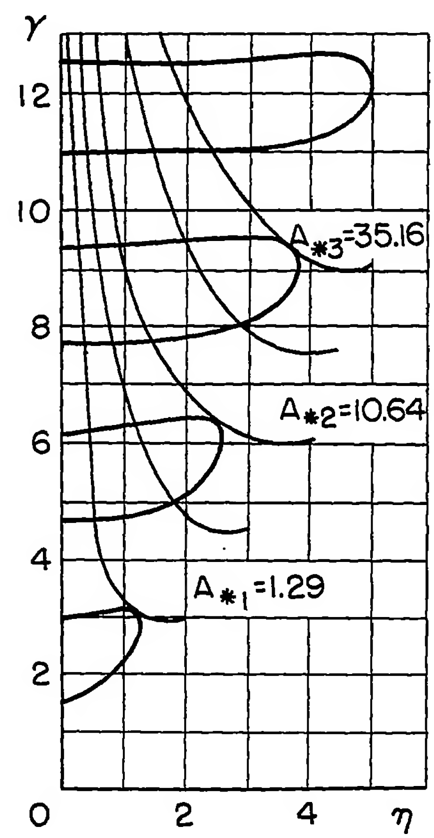

Figure 2.

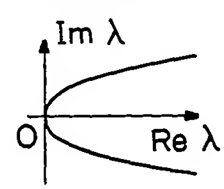

Figure 2 of references 4 and 5.

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